The study of pictorial sequences as a support to the development of algebraic thinking[1]

Neusa Cristina Branco
Escola Superior de Educação, Instituto Politécnico de Santarém
neusa.branco@ese.ipsantarem.pt

João Pedro da Ponte
Instituto de Educação. Universidade de Lisboa
jpponte@ie.ulisboa.pt

Abstract. The purpose of this article is to study the contribution of a teaching unit that introduces algebra through pictorial sequences and problem solving with an exploratory and investigative approach towards the development of the algebraic thinking of grade 7 students. The study focuses on the work of a student, Joana, in different algebraic tasks. Data were collected by means of two interviews, audio recording and a research journal. Prior to the teaching unit, working with a pictorial sequence, the student fails to recognize the algebraic language except in a simple situation. During the teaching unit, the work in the classroom promotes the analysis of different strategies and justifications and the use of the algebraic language. After the teaching unit, the student decomposes the pictures in a sequence and identifies the relation between the order of the picture and the number of its elements, expressing it algebraically. Besides, she uses different strategies to solve equations, showing a significant understanding. She also recognizes the meaning of unknowns and generalized numbers in the symbols that appear in different contexts. Joana's performance suggests that the teaching unit promoted the development of essential aspects of algebraic thinking like the ability to generalize in pictorial sequences, the understanding of symbols, expressions, and equations, and the use of informal and formal strategies for solving equations.

1 Introduction

Since a long time, the teaching of algebra emphasizes algebraic manipulation. Students solve problems that involve the transformation of expressions and the application of rules and procedures. This linguistic-pragmatic approach (Fiorentini, Miorim & Miguel, 1993), leads to a mechanistic and non meaningful learning for many students. As a result, most of them have a very negative view of algebra, especially expressions and equations.

Aiming to improve the development of students' algebraic thinking, Usiskin (1988, 1999) and Kaput (1999) advocate a broader idea of algebra. Blanton and Kaput (2005) characterize algebraic thinking as "a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations

---

[1] Branco, N., & Ponte, J. P. (2012). The study of pictorial sequences as a support to the development of algebraic thinking. *Far East Journal of Mathematical Education*, *8*(2), 101-135.

through the discourse of argumentation, and express them in increasingly formal and age-appropriate way" (p. 413).

In recent years, this is a trend in many countries to emphasize explorations and investigations in mathematics teaching (APM, 1988; NCTM, 2000). In this approach, students build their knowledge from the tasks that they undertake and by discussing their work with their colleagues and with the teacher. This contrasts with direct teaching approaches in which the teacher presents the knowledge in a systematic way and the students' role is to memorize it through repetitive exercises (Ponte, 2005). However, there is still much to be learnt about the ways how explorations and investigations may contribute towards students' learning of specific topics like algebra.

Therefore, the general purpose of this study is to ascertain if a teaching unit for grade 7 students (12-13 years old) based in the study of pictorial sequences contributes to the development of their algebraic thinking and, in particular, to the understanding of variables and equations (Branco, 2008). In this article, we briefly review the literature on issues that have an important relation to this study. Next, we present the aim of the teaching unit, describing its organization, the approach chosen to the classroom and the main goals of the tasks proposed. Then, we indicate the methodology and the results of the work of a student before, during and after the teaching unit. Finally, we discuss the evolution of this student and the contribution that this teaching unit has in the development of algebraic thinking in the student's ability to generalize and in her understanding of the meaning of symbols.

2 Developing symbol sense and algebra learning

The understanding of symbols is of great importance in learning algebra. Arcavi (2006) suggests that, similarly to the teaching of arithmetic, where the main goal is to develop the "number sense", in the teaching of algebra the goal should be the development of "symbol sense". This is a complex knowledge that requires multiple and different experiences throughout several school years. Zorn (2002) addresses students' understanding of the meaning of symbols and stresses the importance that they recognize the mathematical structure of expressions. To this author, symbol sense means "the general ability to extract mathematical meaning from and recognize structure in symbolic expressions, to encode meaning efficiently in symbols, and to manipulate symbols effectively to discover *new* mathematical meaning and structure" (p. 4).

In Portugal, working with algebraic expressions and solving equations are the main foci of the study of algebra at grade 7. Solving equations requires the understanding of the meaning of the equal sign and the symbol-letter as the unknown and working with algebraic expressions requires the understanding of the symbol-letter as a generalized number. According to Kieran (1992) students may have a procedural or structural conception of algebra. The procedural conception refers to the ability to carry out arithmetic operations, substituting values and obtaining numbers, whereas the structural conception refers to of the ability to carry out operations using algebraic expressions instead of numbers.

When solving linear equations, students may follow procedural or structural strategies. Kieran (1992) states that when learning algebra, one must establish connections between algebra and arithmetic in order to develop students' ability to use both perspectives, recognizing the advantages of using one or another, according to the task at hand. In beginning algebra, Kieran (2006) mentions three approaches for solving equations: (i) intuitive approaches, which include using number facts, counting strategies, and cover-up strategies; (ii) trial-and-error substitution; and (iii) the formal approach. Also very used is the balance model, based in the image of equal weights in both sides of a scale (van Ameron, 2002).

In the beginning of the study of algebra, variables are also used as generalized numbers and in functional relationships. According to Ursini and Trigueros (1997), dealing with the different uses of variables requires the following abilities: (i) to recognize their role in a given situation; (ii) to operate with variables; and (iii) to use them in order to symbolize a problematic situation. The concept of variable that students develop depends on the approaches to algebra that they experienced before. In the teaching unit of this study, the tasks proposed include situations that promote students' understanding of different uses of variables.

The development of the ability to generalize is one of the fundamental aspects of algebraic thinking. The study of relations in pictorial sequences constitutes a privileged path to develop that ability. For example, searching regularities and formulating generalizations in a variety of situations should be part of the students' experience since elementary school (Ponte, 2005). Kaput (1999) presents an example of a task regarding the analysis of relations that suggests the use of symbols. In his perspective, this type of task (i) encourages students to work comfortably with symbols, without the reference to numbers and (ii) allows students to experiment mathematics emphasizing understanding. He claims that students should develop from an early age the ability to identify and describe relations, as well as to continue a certain sequence and to create new ones according to some rule. MacGregor and Stacey (1993) also suggest that, on the study of algebra, it is essential the ability to understand a relationship and, afterwards, represent it using algebraic language.

In the present study, students work with repeating and growing sequences (mainly with pictorial sequences) and, later, with equations. Several studies indicate the importance of working with this type of sequence. For example, Mason (2008) suggests that pictorial sequences are rich contexts for generalization. In addition, English and Warren (1999) suggest that the work with these sequences constitute an opportunity to work with symbols, namely through algebraic simplification and analysis of equivalent expressions.

Given some terms of a pictorial sequence, many students can describe how to draw other terms. Some of them can even indicate and justify how will be a further term of a sequence (Mason, 2008). For example, Bishop (1995) identifies, students at grade 7 and 8 using the "model and count" strategy in which they represent the asked order of the picture and count the number of objects that constitute it, determining, in that way, the correspondent term of the numeric sequence. English and Warren (1999) report that students between 12 and 15 years old use an "additive strategy" in which they observe that from a picture to the next one there is a constant increment. However, as Moss, Beatty, Barkin and Shillolo (2008) indicate, the use of this strategy does not lead necessarily to an understanding of the relationship between the order and the number of objects that constitute the picture. Such identification of the general term is a fundamental feature of the development of algebraic thinking (Kieran, 1992) that must be fostered in middle school students (Matos & Ponte, 2008). Kaput (2008) indicates that the work with sequences prepares to other forms of mathematical generalization and promotes the use of syntactic aspects of algebra.

Students can also use different strategies to represent relationship (Mason, 2008; Moss, Beatty, Barkin & Shillolo, 2008). Doing so, contributes to develop their capacity of interpreting symbolic expressions (Mason, 2008) and to identify equivalent expressions. Carraher, Martinez and Schliemann (2008) stress that "different ways of 'visualizing' a pattern are tantamount to different conceptualizations that may lend themselves to different algebraic expressions" (p. 13).

3 The teaching unit

This teaching unit aims to contribute towards the development of the algebraic thinking of grade 7 students, promoting their ability to generalize and to understand the algebraic language. The class has 15 students, some very good, some regular and some very weak. For example, only 9 students never failed any previous year and. 2 students, who failed the previous year, are repeating grade 7. Except for these two students, this is the first time that students work with the algebraic language. In particular, we seek that students develop the ability to represent relationships symbolically and to work with algebraic expressions with understanding. Most of the proposed tasks have an exploratory and investigative nature.

In this unit, the exploration of pictorial sequences (repeating and growing) constitutes the starting point for the study of algebra, as indicated in NCTM (2000) and also in recent Portuguese curriculum documents (ME, 2007). This approach tries to take full advantage of the potential of tasks involving pictorial sequences (Moss, Beatty, Barkin & Shillolo, 2008) and aims to develop students' ability to generalize and to understand the meaning of symbols in different contexts (Arcavi, 2006; Zorn, 2002). The students solved exploratory tasks, discussed different views regarding the terms of the sequences, shared different strategies to analyze the relations and compared results and justifications in the classroom. This kind of work contributes to the understanding of symbolism and promotes the recognition of how it is used in different contexts (Matos & Ponte, 2008).

Before the unit the students studied geometry, measurement and rational numbers, topics such as similar triangles, area, volume, and angle measurement, operations with integers, fractions and decimals. The unit begun in February with a set of tasks involving the study of sequences and then moves on to working with equations (Table 1). Besides the ten main tasks (that are represented ahead by "task B*i*", with $1 \leq i \leq 10$), some exercises and problems from the textbook used in the school were also solved by students.

The weekly mathematics schedule of the school has 2 double periods (90 minutes) and a single period (45 minutes). The teaching unit went for 7 weeks, during as carried out in a total of 35 periods, as indicated on Table 1.

Table 1 Topic distribution and assessment moments

Topics	Tasks	Number of classes (45 minutes)
Sequences	1, 2, 3 and 4	13
Sequences and algebraic expressions	5 and 6	7
Equations	7 and 8	3
Equations and problem solving	9 and 10	8
Assessment moments		4

Each task was the starting point for different learning opportunities, in individual and small group work and in whole class discussions. The work with sequences is a rich context for the emergence of different strategies from the students. However, it is necessary that the questions proposed are diversified and stimulate the emergence of different views, leading the students to find an underling rule and to represent it using algebraic language. This happens when the tasks have an exploratory nature, including open questions and diversified contexts. In the first part of this unit, most tasks involved pictorial and numerical sequences. In a second part, the students worked with equations. This work was based in word problems that challenged students to interpret the situations and to define the unknown and the condition, promoting the use of informal strategies to help them to understand the formal methods.

The tasks were developed generally in groups of two students and the teacher played the role of an advisor, trying to get students involved in the mathematical activity and following their own strategies. When guiding the work of the students, the teacher asked questions in order to help them to clarify their thinking. The exploratory and investigative character of the proposed tasks had a strong influence on the dynamics of the class, since its open nature allowed the students' use of different strategies, favored the discussion of ideas and the mobilization of abilities and previous knowledge, and involved all students in mathematical activity.

The tasks designed to promote the development of students' ability to generalize and use of algebraic language in different contexts, offering progressively more complex situations whose solution benefits from the mobilization of knowledge promoted by the work carried out in previous tasks.

The first four tasks refer essentially to the study of sequences. Task 1 involves repeating sequences. The aim is to analyze and to describe the sequences and to formulate generalizations, namely using knowledge about numbers and multiples. Task

2 presents pictorial growing sequences, in which, in some cases, the general term can be represented by a linear algebraic expression. Students may analyze the transformation that occurs from a term to the following or explore the relationship between the number of elements of a term and its order. Hence, students are asked to identify a rule that they may describe in natural language or they may represent by an algebraic expression. Task 3 assumes a more investigative nature, requiring students to interpret and explore regularities in a problem to answer the situation. On task 4 students are called to analyze, describe, and generalize several numerical sequences from a numerical scheme, which is constituted by numbers from 1 to 20 with an intriguing disposition and that may continue indefinitely. On tasks 5 and 6 there are again linear pictorial sequences which a functional rule can be expressed by an expression of the type $an + b$ In those tasks the possibility that several algebraic expressions could represent the same relationship is explored, leading students to analyze equivalent algebraic expressions.

Tasks 7 and 8 mark the beginning of the study of equations. The first intends to help students to associate to the equal sign a meaning of "equivalence". Task 8 allows the reinforcement of that meaning through exploration of a scale, in which an unknown weight can be determined by an intuitive method or by solving an equation. The students may establish relationships between the description in their natural language and the algebraic language that represents the balance situation and between the intuitive determination of the weight and the final resolution of the equation. This contributes to a meaningful understanding of the equal sign as representing equivalence and of the equivalence principles and practical rules. Solving equations involved working with symbols, thus mobilizing the knowledge developed by working on the tasks on sequences, namely determining equivalent algebraic expressions. Tasks 9 and 10 present several problems which solution can be obtained by solving an equation. Students, initially, tried to solve the problems using their own strategies, without using algebraic language. This situation was explored in order to compare their informal strategies and the solution of the equation that they indicated in each problem.

When most students had done significant work in the task, there was a whole class discussion. This constitutes an important moment for the students' learning process (Stein, Engle, Smith & Hughes, 2008), generated moments of intensive participation from students since they all pretended to share their strategies and to present their conclusions. Questions are posed to the whole class, and every student can participate in answering them. The major conclusions are achieved during the discussion and with the contribution of all students. In working with sequences, it was during whole class discussions that equivalent algebraic expressions first appeared, suggested by different students. That created an opportunity for students to work with symbols doing algebraic simplification. When manipulating expressions, there was a concern to not forget the meaning of the expression in the context. The comparison of equivalent algebraic expressions allowed the students to identify the properties to use, such as the distributive law, and the algebraic operations to be performed. The discussions promoted by the equation tasks constituted an important moment of analysis of the representation in algebraic language and of the students' informal and formal strategies of solving equations. Contrasting strategies enabled an understanding of symbol-letters as unknowns and of the interpretation of conditions and of the algebraic language that could be verified in students when, in other situations, they mobilized that knowledge and were able to use it in their own strategies. In the end of the lesson it was made a summary of the concepts and notions discussed.

4 Research methodology

Given its goal, this study follows a interpretative paradigm (Erikson, 1986). We use a case study methodology to analyze in detail the strategies students follow, their understanding of the algebraic language, and the evolution that they show at the end of the unit. Several students were studied but, in this article, given space constraints, we only consider the case of Joana, a student from the grade 7 class where the unit is taught. The first author is the teacher of the class how had at the time 4 years of experiment. This study provides her an opportunity to investigate her own professional practice in a deep and structured way (Ponte, 2008). This situation also allows her to observe in depth the unfolding of the teaching unit and its contribution to students' learning.

Joana starts the school year as a 12 years old girl. She likes to help her classmates, but, when she does not agree with them, she defends her position with vigor. However, she shows a lack of self-confidence and she is easily distracted. She sees herself as an average student and she considers her performance irregular – sometimes she studies a lot and sometimes she barely studies. During the teaching unit she displays a mixed performance and some difficulty in concentrating in the tasks proposed. In exploratory tasks she has an active participation and shows a strong ability to elaborate strategies and propose solutions for problems.

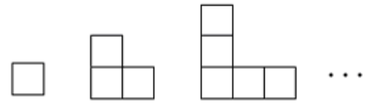
The data collected is essentially descriptive. One of the tools for collecting data is a research journal elaborated by the first author where she recorded her participation and observation of classes. In addition, documents made by the students during the teaching unit were also collected. These two types of document provide means to know the difficulties of the student, her generalization and problem solving strategies, the discussion generated by them, and her understanding of the algebraic language. The major data sources for the case study of Joana are the interviews made individually two weeks before and two weeks after the teaching unit. The two interviews were based on the resolution of a mathematics task consisting of several issues related to the analysis of a pictorial sequence, solving problems with unknown quantities and solving equations. All these topics were developed in the teaching unit through the presented tasks. These interviews give very rich material to analyze the difficulties and strategies as well as the evolution stimulated by the teaching unit. In this article we only take in consideration some of the tasks that Joana answered in those two interviews, trying to identify her generalization ability and her understanding of the meaning of symbols before and after the unit. We also present some classroom episodes from whole class discussions where Joana intervened, showing important moments of her work with algebraic ideas. The tasks identified with A were proposed in the first interview, the tasks identified with B were carried out in the classroom during the teaching unit, and the tasks identified with C were part of the interview made after the teaching unit. The dimensions of analysis are the ability to generalize and the understanding of the meaning of symbols before, during and after the teaching unit.

5 Joana's algebraic thinking before the teaching unit

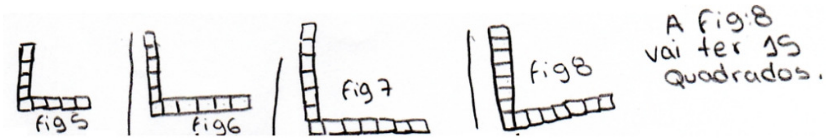
5.1 Ability to generalize

Task A1 presents a sequence of pictures:

Task A1. Observe the following sequence of pictures made with squares:

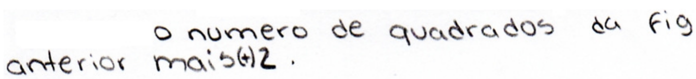


This task includes several questions concerning finding near and distant terms and the general term. To determine near terms (for example, the next term) and not so near terms (for example, the 8^{th} term), Joana explores the relationship between the number of squares of consecutive pictures. Given a known picture, she adds two squares, one at each end of both “arms”:[2]



Picture 8 will have 15 squares.

In these two questions, Joana follows an additive strategy. She also follows an additive strategy in generalizing the relationship between the number of squares of a picture and the order of this picture on the sequence:


o numero de quadrados da fig anterior mais(+)2.

The number of squares from the previous picture plus (+) 2.

Joana always adds two squares to a picture to obtain the next one. That is, her generalization has a descriptive nature and is based on the knowledge of the previous term.

In this way, in the first interview, Joana generalizes a linear pictorial sequence using an additive strategy. She compares the form of consecutive pictures and recognizes that from one picture to the next, it is necessary to add two elements. This strategy allows her to indicate the number of elements of a picture knowing the number of elements of the previous one. Nevertheless, if only the first terms of the sequence are known, this strategy is not adequate to know the number of elements a picture on a very distant position.

5.2 Understanding the meaning of symbols

At the time of the first interview, the student had not started the formal study of algebra. At this moment, we wanted to know the intuitive meanings that she gave to letters and expressions. By knowing these meanings, we may then understand the meanings that she gives to symbols, how she uses them, and how she works with expressions and equations. Task A2 presents a problem where some data is represented by a symbol:

---

[2] Side boxes contain the translation of what students wrote in their work.

Task A2. Ana and Miguel are brother and sister and they decided to count the amount of money that each one has in their piggy bank. Miguel has 5€ more than Ana. If Ana has x€, what can you say about the amount of money Miguel has?

Joana does not give meaning to the data related to Ana's amount of money, represented by x. She does not interpret the letter as a generalized number and she does not perceive this data as information that she can use to represent Miguel's amount of money. She tries to obtain a numeric value or to give instructions related to that amount of money using the only number that was given on the problem:

Joana - I think that he has 25 euro.

Teacher – Why 25?

Joana - That I don't know.

Teacher - So… Why do you think he has 25?

Joana - Because 5 times 5 is 25.

(…)

Joana - He has 5 euro more than her.

Teacher - Yes, he has 5 euro more than his sister. So what? Can you tell anything regarding his money? Besides that?

Joana - I don't think so.

Joana only refers to what is explicitly mentioned in the problem. She wants to give a numerical answer but she considers that impossible because there is not enough information in the problem.

The next task presents two equations. It asked the students to solve them, explaining the strategy that they followed:

Task A3. Determine the value of x in each one of the expressions: $5 + x = 18$ $2x + 3 = 15$

In the first equation, Joana interprets x as the symbol that is in the place of a number. To determine it, she uses the inverse operation and then writes the solution besides the unknown (what may be seen as an informal representation):

$$5+\underset{13}{x}=18 \qquad \begin{array}{r} 18 \\ -5 \\ \hline 13 \end{array}$$

In the second equation Joana does not give a correct meaning to x nor to the expression $2x$, which is related to powers, a topic recently taught in the mathematics class. She cannot explain what she pretends and finishes saying that she does not know what to do:

> Joana - We have to place here a power of a number.
> Teacher - How?
> Joana - A power. I don't know.
> Teacher - How do you understand that expression? What do you think it is?
> Joana - There are two missing.
> Teacher - Missing two what?
> Joana - I don't know if it is a power here, but if it were, it was easier.
> Teacher - So?
> Joana - I don't know how to do.

In task A2, Joana cannot interpret the letter in the expression as data given of the problem that can be used to find the answer. The student interprets the symbol-letter as an unknown in the first equation of task A3 but shows difficulty in the second equation of the same task. That difficulty is related to the understanding of the expression ax. In this way, the student is able to give a meaning to the letter x when it is isolated, but she cannot do the same when the letter x is included in an expression.

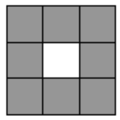
6 Algebraic thinking inside the classroom

6.1 Development of the ability to generalize

Task B6 was done in the classroom. Prior to this task, as described above, students worked with pictorial sequences and sought regularities in problems. So, they had already expressed generalizations in algebraic language and analyzed equivalent algebraic expressions. This task was intended to help students to enhance their ability to generalize based on the analysis of pictorial sequences and understanding of algebraic language. It presents several algebraic expressions related to the number of grey squares of pictures included in a sequence.

It is possible to use different strategies to solve this task. The students may simplify each expression, based on previous knowledge about expressions learnt on previous tasks involving sequences, figuring out which expressions are equivalent, and then relating them to the sequence. They also may analyze the structure of the terms and write a new expression. Afterwards, they may simplify the expressions given and indicate which are equivalent to their expression. Joana does this, but she also discusses the conclusions reached by her colleagues. Thus, she has the opportunity to address the sequence in several ways, showing understanding of the different expressions that arise following diverse methods.

Task B6[3], Question 4. Consider the following pictures:



Picture number 1

Picture number 4

Identify the formulas that can be used to calculate the number of grey squares in any picture (the letter C represents the number of grey squares and N represents the the number of the picture). Explain your choices.

$C = 2N + 3(N + 1)$

$C = 5(N - 1) + 8$

$C = 8 + 5N$

$C = 3(2N + 1) - N$

Going to the blackboard, Joana presents to the class her interpretation of the sequence. She explores the properties of the pictures, decomposing them, and verifies that there is a set of five grey squares with the form ⊏, that is repeated as many times as the order of the picture. Finally, she describes a relationship that she cannot relate to the given expressions. The teacher asks some questions to Joana, to help her to clarify to other students the meaning of the algebraic expression $C = 5N + 3$ that she writes to represent the relationship. During the discussion she indicates to the other students her interpretation of the meaning of N:

Teacher - How many little squares are there in each C?
Joana - 5.
Teacher - How do I write that in a formula?
Joana - 5 times N. 5 times the position number of the picture.
Teacher - 5 times N.
Joana - Plus 3.

The students keep exploring the sequence and try to decompose the pictures in order to establish a relationship between the number of grey squares in each picture and the given expressions. After analyzing the expression proposed by Joana, they concluded that the expression $C = 8 + 5N$ could not represent the number of grey squares of any picture in the sequence.

[3] Adapted from Wijers, Roodhardt, van Reeuwijk, Burrill, Cole & Pligge (2006).

Other students present a decomposition of the pictures of the sequence that allow them to conclude that the expressions $C = 3(2N + 1) - N$ and $C = 5N + 3$ represent the relationship between the number of squares of a picture and its order in the sequence and that the expression $C = 8 + 5N$ does not represent that relationship. Given the difficulties that most students show during the exploration, the interpretation of the other expressions is done with all the class. Finally, the students reflect about the possibility of doing operations in their algebraic expressions in order to verify analytically if all the expressions represent the same relationship. One of them indicates that it is not possible to perform operations on the expression $C = 5N + 3$, proposed by Joana, in order to simplify it. Therefore, it is decided to simplify the other expressions to obtain the same simplified expression for all of them. As a whole class, the students interpret the meaning of parenthesis and proceed, individually, to simplify the expressions $C = 2N + 3(N + 1)$, $C = 5(N - 1) + 8$ and $C = 3(2N + 1) - N$. For each expression, a different student goes to the blackboard to present all the operations he/she did and the class concludes that they are all equivalent, as verified before by the decomposition of the pictures of the sequence. Several students, including Joana, participate in the discussion, for example, regarding the expression $C = 5(N - 1) + 8$:

> Andreia - [$N - 1$] Is the position number of the picture less 1.
>
> Joana - The 5 becomes from the □ [referring to the number of squares of the picture with the form □].

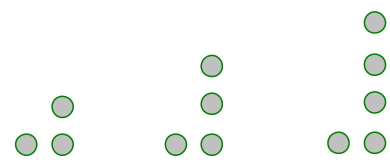
Thus, the exploration of the pictures in different ways contributes to a better understanding of generalizing and fluency in manipulating algebraic symbolism.

The unfolding of this activity reinforces the notion that the study of pictorial sequences and its generalization allows the development of the understanding of equivalence between expressions (English & Warren, 1999). Joana's generalization strategy is primarily based on the analysis of the properties of the pictures of the sequence. The student decomposes a picture and identifies the invariant parts and the parts that are related to the order of the picture. The expression obtained by Joana, and that deserved the consensus of the other students, was then compared to the given expressions, allowing to verify those that were or were not equivalent. The equivalence between two expressions is related to the fact that both of them represent the same relationship, which gives sense to the formal analysis of expressions based on the properties of operations.

6.2 Development of the understanding the meaning of symbols

Task B2 provides a first opportunity for these students to use a symbol-letter to represent a number, formulating an algebraic expression to describe a generalization.

Task B2, Question 1. Observe the sequence of pictures:



a) Draw the next picture of the sequence.
b) Draw the 7th picture of the sequence. How many balls has the picture?
(…)
f) Describe how to construct any picture of this sequence.
g) Write an expression that represents the number of balls of a picture in any position.

The students determine the number of balls of the 4th term, adding a ball to the previous term. To determine the number of balls of other near terms, such as the 7th term, some students still use an additive strategy but others, such as Mariana and Diana, analyze the structure of each term and relate to its constitution to its order:

SE A 1º FIG. TEM 2 BOLAS DE BASE E UMA EM CIMA, A FIG. 2 COMO SE PODE VER TEM 2 BOLAS DE BASE E 2 PARA CIMA E A 3º FIG TEM 2 BOLAS DE BASE E 3 PARA CIMA APLICA-SE O RACICIORO PARA A FIG.:

Mariana e Diana, Q1.b)-T2

If the 1st figure has 2 balls at the base and one upwards, the figure 2, as can be seen, has 2 balls at the base and 2 upwards and the 3rd figure has 2 balls at the base and 3 upwards. One applies the reasoning to the figure.

This strategy appears most effective in determining a more distant point and this issue is discussed with the whole class when the students present their answers to the question "Describe how it is constructed any figure of this sequence". Susana still follows an additive strategy but she recognizes it does not work when she is asked about a distant order term:

> Susana - Teacher, I put this: There are always two balls at the base and an increase one upwards according to the sequence.
>
> Teacher - Yes, but this is based on the previous.
>
> Susana - Yes.
>
> Teacher - So, if I ask you to fifty you have to draw all up to forty-nine and then add one more ball?
>
> Susana - Oh, teacher! I have two down and then I increase on top.
>
> Teacher - Why do you increase one on top?

Susana - Based on the previous.

Teacher - If I ask fifty you will draw...

Susana - No.

Teacher - How do you draw, for example, the figure twenty-five?

Rafaela - I make two balls at the base and then twenty-five in the top.

Rafaela relates the total number of balls with the order of the term but does not addresses the disposition of balls. Joana and Catarina establish this relationship in a general way:

A fig:.
acrescenta-se 2 ao nº da fig.

Joana e Catarina

The pict: One adds 2 to the number of the picture.

The next question asks for an expression of the general term, but the students do not understand the meaning of the word "expression". So, the question is discussed with the whole class and this enables the establishment of a general the relationship between the term of the numerical sequence of the number of balls and its order. For example, Susana who previously used an additive strategy, now suggests this relationship:

Teacher - I have a figure in any position, how do I calculate the number of balls that this figure will have?

Susana - It is the number of balls in vertical, the number of the figure, and then one adds to two balls to make up the base.

From this exploration, a range of examples emerge:

figura 10 → 2+10
figura 30 → 2+30
figura 400000 → 2+400 000

Xico e Ricardo

For the general term, Susana proposes, then, that, as we do not know the order of the figure, we represent it by a symbol, a question mark:

Susana – Teacher, Oh, I know. It is two and a question mark.

Teacher – It may be. Two and a question mark. What does this question mark represent?

Diana – The number of balls that we have to add.

Susana – The number of the figure because we do not have the number.

The symbol that Susana indicates represents an unknown number that can take several values. The students come thus to a general expression for determining the number of balls which constitute any figure. Then, the teacher suggests that other symbols may be used, according to the situation, with the same meaning. Belmiro, a student who attends grade 7 for the second time, immediately suggests the use of the letter *x*. Some students suggest other letters, such as *b*, *c*, *h*, or *n*:

figura ? —> e+?
n —> e+n

Xico e Ricardo

The discussion involves several students promotes understanding of the generalization and gives rise to the use of algebraic language to express it.

Another example of students developing the meaning of symbols concerns word problems. Solving such problems, in an initial phase, students mostly use informal strategies. In the discussion, some connections between the students' intuitive representations expressed in natural language and the algebraic representation of the situations are established. This motivates the students to use algebraic language, as, for example, in the following problem:

> Task B9, Question 4. During the weekend José goes for bicycle rides. On Sundays he does 6 more kilometers than on Saturdays. He does a total of 38 kilometers. How many kilometers does he do on each day?

The students think a little bit about the question. Then, the equation that represents the problem is written on the board, with contributions from several students, including Joana. One student suggests to use *x* to represent the number of kilometers and the following discussion takes place:

Teacher - *x* is the number of kilometers that he rode… When?

Diana - On Saturday.

Teacher - Very good. And how do you represent the number of kilometers that he rode on Sunday?

Belmiro - *x* plus 6…

Teacher – That's the number of kilometers he rode on Sunday. Let's write an expression to determine the total. How do we do that?

Joana - *x* plus *x* plus 6.

Francisco - That's equal to 38.

Teacher - What he rode on Saturday plus…

Joana - what *he* rode on Sunday, *x*…

Francisco - Plus *x* plus 6 it's equal to 38.

Joana gives significant contributions concerning the elaboration of the expression and to the establishment of its meaning. In the subsequent discussion she adds that on Saturday José rides 16 kilometers and on Sunday he rides $16+6$, which is 22 kilometers.

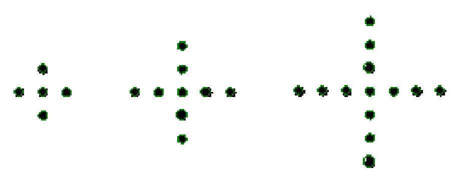
During classes, as it also occurred in other studies (e.g., Bednarz & Janvier, 1996; van Ameron, 2002), Joana and other students show some initial difficulty in using equations to solve problems. However, the whole class discussions create many opportunities for different students to make contributions that help interpret ting the algebraic language and using it with understanding. In this problem, for example, Joana correctly understands the unknown and gives meaning to the algebraic representation of the problem, showing progress in understanding the meaning of the symbol-letter and in using the algebraic language.

7 Joana's algebraic thinking after the teaching unit

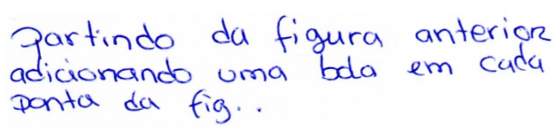
7.1 Ability to generalize

Task C1 from the second interview presents a linear pictorial sequence. We wanted to identify the strategies followed by Joana in order to determine the number of elements of a near term (the next) and of a not so near term (the 10th) and to determine the relationship between the order of a picture and the number of dots that constitute it.

Task C1. Consider the following sequence of pictures:



This task has three questions concerning to find near and distant terms and the general term. In the first question, Joana analyzes the number of dots of consecutives pictures and identifies a regularity in the formation of a picture of the sequence based on the previous picture. She adds four dots to the 3rd picture, one on each end, and obtains the 4th picture:



Starting from the previous picture adding one ball in each end of the picture.

To determine the number of dots of a picture in a distant position, the 10th term, Joana does not follow an additive strategy, as previously. She observes the different pictures of the sequence and looks for a regularity in their geometric properties. Through the description of the drawing she identifies a central dot in all pictures and four parts with a number of dots equal to the order of the picture:

Teacher - How many dots does the 10th picture have?

Joana - It has 40.

Teacher - Why?

Joana - No.

Teacher - So?

Joana - I don't know.

Teacher - So, if you wanted to draw. How would you do it? Maybe it will help you to know how many dots it has.

Joana - I would draw the first dot and would add… 10.

Teacher - How do you do it then?

Joana - I would draw a dot…

Teacher - Yes.

Joana - And I would add 10… 10 dots on each arm.

Joana clarifies her reasoning and finds out her mistake when she describes the method to draw the 10th picture. She relies on the analysis of the properties of each picture on the sequence and establishes a relationship between the number of dots of a picture and its order on the sequence.

She presents the generalization of this situation in algebraic language. She represents by *n* the order of the picture on the sequence and by *t* the total number of dots:

Joana - It is the position number of the picture times 4, plus something.

Teacher - Plus what?

Joana - I think it is plus 1, which is the central one.

$$n \times 4 + 1 = t$$

In the second interview, Joana clearly shows an evolution on her ability to explore and identify relations in sequences (not only between consecutive pictures), which allows her to indicate a relationship between the order of the picture and the number of its elements using algebraic symbolism. Joana identifies this relationship based on the analysis of the properties of the pictures in the sequence, like she does in task B6, previously reported. In the classroom, from tasks B2 to B6, students were encouraged to express in algebraic language the relations that they found and that they explained in natural language. Whereas in the first interview the student exclusively uses natural language to express the relationship, she now expresses herself in an algebraic language, using symbols-letters as generalized numbers.

7.2 Understanding the meaning of symbols

Task C2 requires a representation for the perimeter of the triangle that satisfies certain conditions and is used to verify if Joana uses algebraic language and what meaning she gives to symbols.

Task C2. The three sides of a triangle have different lengths. The second side has three centimeters more than the first side and the third side has twice the length as the first side.

Joana is asked how to represent the perimeter of the triangle. She reads the problem, draws a triangle and analyzes the conditions given:

Joana - The perimeter is equal to the sum of all sides.
Teacher - Exactly. Very good.
Joana - How do I know how much is the first one? There's no way to find out.
Teacher - Do you have any idea about what you can do?
Joana - It can be several.
Teacher - Several what?
Joana - Several centimeters.

Joana verifies that the length of the first side (the smallest side of the triangle) can take several values (being a triangle, the length of the first side has to be bigger than 1.5 units of length). She represents by n the value of the length of that side and expresses the length of the other two sides as a function of it, by the expressions $n+3$ and $n \times 2$. Afterwards, she elaborates an algebraic expression for the perimeter of the triangle:

Joana – *n*. Plus 3. No. *n* plus *n* plus 3. Plus *n* times 2.
Teacher – Write.
Joana – I think that is.
Teacher – Write.
Joana – *n*. That is the first side.
Teacher – Very well.
Joana – Plus… *n* plus 3. That is the second side.
Teacher – Hum, hum.
Joana – Plus *n*… Or *n* plus *n*, or *n* times 2. Do I put times 2 or just 2?
Teacher – What you want.
Joana – So I put times 2.

$$n+n+3+n\times 2$$

Joana translates the relationships of the problem given on natural language into algebraic language, showing an understanding of symbols in this context and the usefulness of using them. In the first interview Joana could not express a problem in algebraic language. In the teaching unit from task B8 to task B10, the students represented different situations in algebraic language. For example in task B9 – Question 4 of the teaching unit, Joana gives some contributions to the use the algebraic language in solving word problems and to the clarification of the meaning of the unknown. By the end of the teaching unit, it is possible to verify her evolution in using algebraic language to represent problems by herself.

In the classroom, the equations to solve always originated in word problems. In the beginning, students used informal strategies to find the value of the unknown and progressively started to use formal strategies that they compared with their informal strategies. In more complexes equations they needed to used the algebraic properties and operations. In the second interview, we checked if Joana can also solve equations without a real-life context and if she recognizes the adequate meaning of algebraic language that she did not recognize in task A2 of the first interview. Task C3 presents six equations that had the purpose of identifying the meaning the Joana give to algebraic notation and her strategies in solving equations and of finding out her understanding of algebraic language.

Task C3. Solve the following equations:

a) $12 = x + 9$	b) $8x = 4$
c) $10 + x = 4$	d) $-3 - x = 0$
e) $8 + 3x = 20$	f) $3(x-7) + 10 = 2x - 3$

Up until equation d), Joana follows a procedural strategy to mentally determine the solution. She retrieves her knowledge of properties of numbers in order to have an intuitive idea of the unknown number that satisfies the condition. Afterwards, she uses formal strategies to solve each equation and determine its solution. Her answer to question b) is an example of this kind of thought. The student indicates, based on her knowledge of properties of numbers, the solution of the equation, and then, solves it through the procedures usually followed in class, dividing both sides of the equation by the coefficient of the x term:

Joana - Each x will be 0.5. I think.

Teacher - How did you do?

Joana - I don't know.

Teacher - So!?

Joana - Each two x has to be one. I only need to see how much one x is.

b) $\frac{8x}{8} = \frac{4}{8}$

$x = 0{,}5$

Joana shows some insecurity in solving equation f), as it happened throughout the whole interview. At the beginning she refers that she does not know how to solve these equations. Nevertheless, she gives meaning the algebraic expressions:

> Joana - I have $3x$ and I have -7. Ah! It is 3 times what is inside of this.

Joana correctly interprets the expression $3(x-7)$ and tries to solve the equation. She tries to find alike terms in both sides of the equation and follows the balance model, but the terms that she refers to are not alike, making impossible to use that strategy:

> Teacher - So, what is the first thing you think you have to do?
>
> Joana - If I join the 3 is zero. Yes, the 3, I don't know if it is correct because one is positive and the other is negative.
>
> Teacher - Do you think they have exactly the same role in both sides of the equation?
>
> Joana - No.
>
> Teacher - So?
>
> Joana - Here 3 is… 3 times all this. And here isn't… So I can't take out a 3. I can take out an x on each side. Can I?
>
> Teacher - What will happen?
>
> Joana - I will be without x here [on the first side of the equation] and I will have only one x here [on the second side of the equation].
>
> Teacher - Is it?
>
> Joana - I think it is.
>
> Teacher - Are you taking out one on each side?
>
> Joana - Ah, I can't take this one [from the first side of the equation].
>
> Teacher - So?
>
> Joana - Because there is a 3 on it.

Joana shows an understanding of the meaning of each side of the equation and she realizes that the procedures that she wants to follow are not adequate. After that, she brings back the initial interpretation of the expression $3(x-7)$ and applies the distributive property. After doing this she compares the expression in each side of the equation and identifies the terms with x – on the first side of the equation, the 3x term, and, on the second side, the 2x term. She uses the balance strategy and, without writing the procedure, she takes away 2x on each side of the equation. Finally, she passes all the independent terms to the second side of the equation and proceeds with solving the equation, correctly determining its solution.

Using the terms of Kieran (2006), we may say that when solving these equations Joana follows intuitive strategies, based on her knowledge of numbers and their properties and when solving the last equation she follows formal strategies. She uses the transposition procedure, that is, changes a term from one side of the equation to the other changing the sign, and the procedure involving doing of the same operations on both sides of the equation. She also cancels equal terms on both sides of the equation. The fact that she combines different strategies to solve an equation shows a significant understanding of equations and of different strategies, formal and informal.

In the second interview, we also see that Joana developed a deeper understanding of the algebraic language. On task C1 she uses, by her own initiative, symbolism and recognizes *n* as a generalized number:

> Joana - I wrote a letter because I couldn't use a number.
>
> Teacher - Why couldn't you use a number?
>
> Joana - Because if I did it, it will be very confusing and you will think, if I put 1 it will be 1×4 and the position number of the picture can't be 1. Because n is any number. An undefined number, I don't know which one.

On task C3 she interprets *x* as an unknown, which means that it represents a value that she does not know but that may be represented. Joana distinguishes between the meanings of the variable on both situations – in one she recognizes the symbol-letter as a generalized number and on the other as an unknown. The following excerpt illustrates this situation:

> Joana - It will be known how much each x is.[on task C3].
>
> Teacher - It will allow you to know the values of each x. So, what about in C2, for example, doesn't that allow you to know how much is *n*?
>
> Joana - It will be 3*n*… No… I mean, I don't know. Maybe it will be possible.
>
> Teacher - So?
>
> Joana - But there is no equal sign here.
>
> Teacher - There is no equal sign.
>
> Joana - No.
>
> Teacher - If there was an equal sign, was it possible then?
>
> Joana - Yes.

Joana clearly understands the difference between an expression that represents a generalization and an equation. She also shows an understanding of the meaning of the equal sign in algebraic language, since the equality of two algebraic expressions (up until now linear) establishes a condition that is true for one given value. On the second interview she interprets the symbol-letter, both as a generalized number and as an unknown. She clearly shows to be conscious about the fact that symbols can have different roles in different contexts, a key element of symbol sense (Arcavi, 2006).

8 Conclusion

This article presents the work developed by Joana, a grade 7 student, in order to understand if a teaching unit based on an exploratory approach enhances the development and the mobilization of students' algebraic thinking. The student was interviewed before and after the teaching unit, solving mathematical tasks that allow the analysis of her performance in two main aspects of algebraic thinking – the ability to generalize and the understanding of the meaning of symbols.

Ability to generalize. In both interviews, Joana identifies relationships in pictorial sequences. In the first interview, she uses additive strategies in order to find a near term and a not so near term and to make generalizations about the sequence that she expresses in natural language. In the second interview, she keeps using additive strategies (to analyze consecutive pictures) but uses also decomposition strategies (to analyze the properties of pictures and establish relationships between the number of elements of a picture and its order in a sequence). She only uses additive strategies to determine a near term. To answer other questions, Joana analyses the geometric properties of the pictures and identifies the relationship between the order and the number of elements of the picture, as the students in the studies of Bishop (1995) and English and Warren (1999) did. She expresses these relationships in algebraic language, with the symbol-letter as a generalized number. In addition, whereas in the first interview she only formulates generalizations in natural language, in the second interview she is able to express such generalizations using the algebraic language. Therefore, Joana shows a significant evolution.

The work in the classroom was based in exploratory tasks, many of them involving pictorial sequences, requiring different sorts of generalizations (to find a near term, a distant term, a general term…). Connections between numerical, visual and symbolic thinking was promoted. Also important was the discussion of students' solutions that led them to contrast their generalization strategies and understandings. Together, these two elements of the teaching unit (the kind of tasks and the classroom way of work) seem to have contributed to the development of Joana's ability to generalize in sequences and her use of the algebraic language to represent such generalizations.

Understanding the meaning of symbols. During the first interview, Joana only recognizes the meaning of the symbol-letter as an unknown in equations such as

$x + a = b$, where a and b are natural numbers. She determines the solution doing the inverse operations, i.e., subtracting $b - a$. She does not recognize that same meaning in an equation that contains the expression $2x$ – an expression that she does not know. In contrast, during the second interview, the symbol-letter appears in problems posed in different contexts and Joana gives it correct meanings in relation to those contexts. She recognizes the symbol-letter as an unknown in the equations that she faced and as a generalized number in expressions that she made herself. When solving the first equations she follows informal strategies to find the solutions in an intuitive way. In the equations of task C3, of the type $ax = b$ and $x + a = b$, she correctly determines the solution and justifies her answer by recalling her knowledge of numbers and of properties of operations. When the characteristics of the equations do not favor the use of intuitive strategies, she uses formal strategies. She follows formal strategies depending on her interpretation of the equation, namely, doing transpositions and using the same operation in both sides of the equation. In these situations she recognizes the mathematical structure of expressions, an important aspect of symbol sense (Zorn, 2002). This understanding of equations and the knowledge of various strategies for solving equations was developed in the classroom, with tasks B8, B9 and B10. The problems that could be solved in several ways, and stimulated translation processes and writing of algebraic expressions, were always supported by work in small groups and whole class discussions.

Regarding the algebraic language, at the end of the teaching unit, Joana shows a very significant understanding of the interpretation of variable as unknown and as generalized number. She clearly describes its meaning depending on the expression or equation, which contrasts with her performance in the first interview.

The power of an exploratory approach. The work developed during the teaching unit introducing the study of algebra is based on the study of sequences and on problem solving that naturally promote the generalization and the use of algebraic language. The evolution shown by Joana strongly suggests that this approach supported her learning. The development of her ability to generalize was enhanced by the exploration of pictorial sequences, as these tasks favor the identification of relationships based on the analysis of properties of pictures. The fact that these generalizations can be formulated in natural language and can be represented by different algebraic expressions, gives meaning to the symbolism used and to the notion of equivalent algebraic expressions

(English & Warren, 1999; Mason, 2008), enhancing the students' understanding of the algebraic language.

Joana shows symbol sense that contributes to a progressive and significant development of her algebraic thinking. She uses algebraic language to represent generalizations and manipulates symbolically the expressions and equations using adequate procedures, showing understanding of symbols (Arcavi, 2006). Most students in this class had an active participation in the classroom discussions and showed an equally significant evolution in their ability to generalize, using algebraic language to represent relationships as well as in working with symbols (Branco, 2008).

Other studies (e.g., Carraher, Martinez & Schliemann, 2008; English & Warren, 1999) have suggested the value of working with pictorial sequences to promote students' ability to generalize and to develop a feeling for the use of the algebraic language. This study is limited to a trial of a teaching unit to a single classroom and presents data of only one student. However, there was nothing special about this class and several other students showed a similar evolution. Therefore, it is reasonable to claim that this study suggests that the beginning of the formal study of algebra, oriented to the understanding of expressions and equations, may benefit from being taught via a teaching unit based in exploratory and investigation tasks, involving sequences and solving word problems, an arrangement that supports a movement from informal strategies to more formal methods. Teaching such unit with positive results has proved to be feasible in the setting of a regular class. In addition, working on these tasks, contrasting strategies and justifying solutions in classroom discussions, seems to favor students' establishing connections between concepts, stimulate their understanding of different types of representations and allow them to change from natural language to the use of algebraic language, supporting the development of their algebraic thinking.

References

APM (1988). *Renovação do currículo de Matemática*. Lisboa: Associação de Professores de Matemática.

Arcavi, A. (2006). El desarrollo y el uso del sentido de los símbolos. In I. Vale, T. Pimentel, A. Barbosa, L. Fonseca, L. Santos & P. Canavarro (Orgs.) *Números e Álgebra na aprendizagem da Matemática e na formação de professores* (pp. 29-48). Lisboa: SEM-SPCE.

Bednarz, N. & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra* (pp. 115-136). Dordrecht: Kluwer.

Bishop, J. W. (1995). Mathematical patterns in the middle grades: Symbolic representations and solution strategies. In L. Meira & D. Carraher (Orgs.), *Proceedings of the 19th PME International Conference* (Vol. 1, pp. 73-79). Recife: PME.

Blanton, M., & Kaput, J. (2005). Characterising a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, *36*(5)l, 412-443.

Carraher, D., Martinez, M., & Schliemann, A. (2008). Early algebra and mathematical generalization. *ZDM*, 40(1), 3-22.

English, L., & Warren, E. (1999). Introducing the variable through patterns exploration. In B. Moses (Ed.), *Algebraic thinking, grades K-12* (pp. 141-145). Reston, VA: National Council of Teachers of Mathematics.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), Handbook of Research on Teaching (pp. 119-161). New York: Macmillan.

Fiorentini, D., Miorim, M. A., & Miguel, A. (1993). Contribuição para repensar a educação algébrica elementar. *Pro-Posições, 4*(1), 78-90.

Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Orgs.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades*. (pp. 5-17). New York: Lawrence Erlbaum Associates.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York, NY: Macmillan.

Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on psychology of mathematics education: Past, present and future* (pp. 11-49). Rotterdam/Taipei: Sense Publishers.

MacGregor, M., & Stacey, K. (1993). Seeing a pattern and writing a rule. In I. Hirabayasbi (Ed.), *Proceedings of the 17th PME International Conference* (Vol. 1, pp. 181-188). Tusukuba, Japan.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades*. (pp. 57-94). New York: Lawrence Erlbaum Associates.

ME (2007). *Programa de Matemática do Ensino Básico*. Retrieved at 12.Jan.2008 from http://sitio.dgidc.min-edu.pt/matematica/Documents/ProgramaMatematica.pdf.

Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008). "What is your theory? What is your rule?" Fourth graders build an understanding of functions through patterns and generalizing problems. In C. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 155-168). Reston, VA: NCTM.

NCTM (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., Engle, R. A., Smith, M., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313-340.

Ursini, S., & Trigueros, M. (1997). Understanding of different uses of variable: A study with starting college students. In E. Pehkonen (Ed.), *Proceedings of the 21st PME International Conference* (vol. 4, pp. 254-261). Lahti, Finland.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Schulte (Eds.), *The ideas of algebra* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (1999). Why is algebra important to learn? In B. Moses (Ed.), *Algebraic thinking, grades K-12* (pp. 22-30). Reston, VA: National Council of Teachers of Mathematics.

van Ameron, B. (2002). *Reinvention of early algebra: The developmental research on the transition from arithmetic to algebra* (Dissertation thesis, Freudhental Instituut, Utrecht). Retrieved at 26.Sep.2005 from http://www.library.uu.nl/digiarchief/dip/diss/2002-1105-161148/inhoud.htm

Wijers, M., Roodhardt, A., van Reeuwijk, M., Burrill, G., Cole, B., & Pligge, M. A. (2006). Building formulas. In *Mathematics in context*. Chicago, IL: Encyclopædia Britannica.

Zorn, P. (2002). Algebra, computer algebra, and mathematical thinking. In *Proceedings of the 2nd International Conference on the Teaching of Mathematics*. Retrieved at 14.Dec.2005 from http://server.math.uoc.gr/~ictm2/Proceedings/pap426.pdf.